Astrometry of an image

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1 Approximate celestial coordinates of an image from some identified stars

Let us assume that some stars appearing in the image have known coordinates (α_i, δ_i), $i = 1 \ldots n$. The pixel coordinates of the same stars are (X_i, Y_i), $i = 1 \ldots n$. Let us call

$$\begin{aligned} a &= (\alpha - \langle\alpha\rangle)\cos\delta, \\ d &= \delta - \langle\delta\rangle, \end{aligned} \qquad (1)$$

and

$$\begin{aligned} x &= X - \langle X\rangle, \\ y &= Y - \langle Y\rangle. \end{aligned} \qquad (2)$$

The objective function to minimize is

$$Q = \sum_{i=1}^{n} [a_i - (Cx_i - Sy_i)]^2 + [b_i - (Sx_i + Cy_i)]^2, \qquad (3)$$

where C and S are the two free parameters of the fit. We need to calculate the following second order moments: $\langle x^2\rangle$, $\langle y^2\rangle$, $\langle xa\rangle$, $\langle xd\rangle$, $\langle ya\rangle$, and $\langle yd\rangle$. The parameters C and S are given by

$$\begin{aligned} C &= \frac{-\langle xa\rangle + \langle yd\rangle}{\langle x^2\rangle + \langle y^2\rangle}, \\ S &= \frac{\langle ya\rangle + \langle xd\rangle}{\langle x^2\rangle + \langle y^2\rangle}. \end{aligned} \qquad (4)$$

The pixel size, s, and the position angle of the Y direction, ϕ, are given by

$$\begin{aligned} s &= \sqrt{C^2 + S^2}, \\ \phi &= \mathrm{atan2}\,(C, S). \end{aligned} \qquad (5)$$

The equations of the coordinate transformation are

$$\begin{aligned} (\alpha - \langle\alpha\rangle)\cos\delta &= s[(X - \langle X\rangle)\cos\phi - (Y - \langle Y\rangle)\sin\phi] \\ &= C(X - \langle X\rangle) - S(Y - \langle Y\rangle), \\ \delta - \langle\delta\rangle &= s[(X - \langle X\rangle)\sin\phi + (Y - \langle Y\rangle)\cos\phi] \\ &= S(X - \langle X\rangle) + C(Y - \langle Y\rangle). \end{aligned} \qquad (6)$$

The inverse transformation is

$$\begin{aligned} X &= \langle X\rangle + \frac{1}{s}[(\alpha - \langle\alpha\rangle)\cos\delta\cos\phi + (\delta - \langle\delta\rangle)\sin\phi] \\ &= \langle X\rangle + \frac{1}{C^2 + S^2}[C(\alpha - \langle\alpha\rangle)\cos\delta + S(\delta - \langle\delta\rangle)], \\ Y &= \langle Y\rangle + \frac{1}{s}[-(\alpha - \langle\alpha\rangle)\cos\delta\sin\phi + (\delta - \langle\delta\rangle)\cos\phi] \\ &= \langle Y\rangle + \frac{1}{C^2 + S^2}[-S(\alpha - \langle\alpha\rangle)\cos\delta + C(\delta - \langle\delta\rangle)], \end{aligned} \qquad (7)$$

2 Alignment of two images

The same procedure can be applied to the alignment of two images. Let us assume that we identify several objects common to both images. The pixel coordinates in the first image will be called (X_1, Y_1), and those in the second image (X_2, Y_2). Let us call

$$\begin{aligned} x_1 &= X_1 - \langle X_1 \rangle, \\ y_1 &= Y_1 - \langle Y_1 \rangle, \\ x_2 &= X_2 - \langle X_2 \rangle, \\ y_2 &= Y_2 - \langle Y_2 \rangle. \end{aligned} \tag{8}$$

We calculate

$$\begin{aligned} C &= \frac{-\langle x_1 x_2 \rangle + \langle y_1 y_2 \rangle}{\langle {x_2}^2 \rangle + \langle {y_2}^2 \rangle}, \\ S &= \frac{\langle y_1 x_2 \rangle + \langle x_1 y_2 \rangle}{\langle {x_2}^2 \rangle + \langle {y_2}^2 \rangle}. \end{aligned} \tag{9}$$

The equations of the coordinate transformation are

$$\begin{aligned} x_1 &= C x_2 - S y_2, \\ y_1 &= S x_2 + C y_2, \end{aligned} \tag{10}$$

or

$$\begin{aligned} X_1 &= \langle X_1 \rangle + C(X_2 - \langle X_2 \rangle) - S(Y_2 - \langle Y_2 \rangle), \\ Y_1 &= \langle Y_1 \rangle + S(X_2 - \langle X_2 \rangle) + C(Y_2 - \langle Y_2 \rangle). \end{aligned} \tag{11}$$

The inverse transformation is

$$\begin{aligned} x_2 &= \frac{1}{C^2 + S^2}(C x_1 + S y_1), \\ y_2 &= \frac{1}{C^2 + S^2}(-S x_1 + C y_1), \end{aligned} \tag{12}$$

or

$$\begin{aligned} X_2 &= \langle X_2 \rangle + \frac{1}{C^2 + S^2}[C(X_1 - \langle X_1 \rangle) + S(Y_1 - \langle Y_1 \rangle)], \\ Y_2 &= \langle Y_2 \rangle + \frac{1}{C^2 + S^2}[-S(X_1 - \langle X_1 \rangle) + C(Y_1 - \langle Y_1 \rangle)]. \end{aligned} \tag{13}$$