1 Approximate celestial coordinates of an image from some identified stars

Let us assume that we have an image far from the celestial poles, with a small field of view, so that sky curvature can be ignored. Some stars appearing in the image have known coordinates \((\alpha_i, \delta_i)\), \(i = 1 \ldots n\), while the pixel coordinates of the same stars are \((X_i, Y_i)\), \(i = 1 \ldots n\). Let us call

\[
\begin{align*}
a &= (\alpha - \langle \alpha \rangle) \cos \delta_0, \\
d &= \delta - \langle \delta \rangle,
\end{align*}
\]

where \(\delta_0\) is the declination of the image center, \(\langle \alpha \rangle = (1/n) \sum_{i=1}^{n} \alpha_i\), and \(\langle \delta \rangle = (1/n) \sum_{i=1}^{n} \delta_i\). The coordinates \((a, d)\) are cartesian coordinates with origin at the average position \((\langle \alpha \rangle, \langle \delta \rangle)\), with \(a\) increasing eastwards and \(d\) increasing northwards (negative orientation). Similarly, we define the pixel cartesian coordinates

\[
\begin{align*}
x &= X - \langle X \rangle, \\
y &= Y - \langle Y \rangle,
\end{align*}
\]

with origin at the average position \((\langle X \rangle, \langle Y \rangle)\), with \(x\) increasing rightwards and \(y\) increasing upwards (positive orientation).

The transformation between pixel and celestial coordinates depends on two parameters, which can be interpreted as a scale factor (the pixel size, \(p\)), and a rotation angle (the position angle of the \(Y\) direction, \(\phi\)). It is more practical to use the parameters

\[
\begin{align*}
C &= p \cos \phi, \\
S &= p \sin \phi,
\end{align*}
\]

so that the coordinate transformation equations are

\[
\begin{align*}
a &= -Cx + Sy, \\
d &= Sx + Cy,
\end{align*}
\]

and those of the inverse transformation are

\[
\begin{align*}
x &= \frac{1}{C^2 + S^2} (-Ca + Sb), \\
y &= \frac{1}{C^2 + S^2} (Sa + Cb).
\end{align*}
\]

The parameters \(C\) and \(S\) have to be determined through the minimization of the objective function

\[
Q = \sum_{i=1}^{n} [a_i - (-Cx_i + Sy_i)]^2 + [b_i - (Sx_i + Cy_i)]^2.
\]

The values obtained for the parameters \(C\) and \(S\) can be given in terms of the second order moments \((x^2), (y^2), (xa), (xd), (ya),\) and \((yd)\),

\[
\begin{align*}
C &= \frac{\langle yd \rangle - \langle xa \rangle}{\langle x^2 \rangle + \langle y^2 \rangle}, \\
S &= \frac{\langle xd \rangle + \langle ya \rangle}{\langle x^2 \rangle + \langle y^2 \rangle}.
\end{align*}
\]
Note that for an image nearly aligned with the celestial coordinates, \( \phi \approx 0, \ C \approx p, \) and \( S \approx 0, \) resulting in \( a \approx -px \) and \( d \approx py. \)

Once the values of \( \langle \alpha \rangle, \ \langle \delta \rangle, \ \langle X \rangle, \ \langle Y \rangle, \ C, \) and \( S \) are known, the celestial coordinates of any pixel position \((X,Y)\) can be obtained by applying Eqs. 2, 4, and 1.

## 2 Alignment of two images

A similar procedure can be applied to the alignment of two images. Let us assume that we identify several objects common to both images. The pixel coordinates in the first image will be called \((X_1,Y_1)\), and those in the second image \((X_2,Y_2)\). Let us call

\[
\begin{align*}
  x_1 &= X_1 - \langle X_1 \rangle, \\
  y_1 &= Y_1 - \langle Y_1 \rangle, \\
  x_2 &= X_2 - \langle X_2 \rangle, \\
  y_2 &= Y_2 - \langle Y_2 \rangle.
\end{align*}
\]

Note that in this case we assume that both coordinate systems have the same (positive) orientation.

The equations of the coordinate transformation are

\[
\begin{align*}
  x_1 &= Cx_2 - Sy_2, \\
  y_1 &= Sx_2 + Cy_2,
\end{align*}
\]

and those of the inverse transformation are

\[
\begin{align*}
  x_2 &= \frac{1}{C^2 + S^2}(Cx_1 + Sy_1), \\
  y_2 &= \frac{1}{C^2 + S^2}(-Sx_1 + Cy_1).
\end{align*}
\]

The parameters \( C \) and \( S \) are given by

\[
\begin{align*}
  C &= \frac{\langle x_1x_2 \rangle + \langle y_1y_2 \rangle}{\langle x_2^2 \rangle + \langle y_2^2 \rangle}, \\
  S &= \frac{\langle y_1x_2 \rangle - \langle x_1y_2 \rangle}{\langle x_2^2 \rangle + \langle y_2^2 \rangle}.
\end{align*}
\]

Note that for two images nearly aligned and with the same pixel size \((C^2 + S^2 = 1)\), the values of the parameters have to be \( C \approx 1 \) and \( S \approx 0. \)