Homework 6: Solution

1. If we change the difference between the proton and neutron mass to $Q_n = 0.129$ MeV, while all other parameters remain the same, the time of freeze-out of the neutron abundance occurs at the same temperature $kT_{freeze} = 0.8$ MeV, and so the neutron abundance freezes out at $n_n/n_p = e^{-0.129/0.8} = 0.851$. The helium abundance, if there were no neutron decays and they all combined to form helium, would then be

$$Y = \frac{2n_n}{n_n + n_p} = 0.92 \; . \tag{1}$$

(note that neutrons would in fact not decay, they would be stable because the difference with the mass of the proton would be less than the mass of the electron).

It would be rather unfortunate if the neutron had a mass so close to the proton mass: almost all the matter in the universe would have turned to helium in the beginning of the universe, and main-sequence stars would not live very long with the very small amount of hydrogen they would have left. The Sun would live for less than 1 billion years and the planet Earth would not have had enough time to sustain life on it for us to be here now.

2. (a) The energy density of photons at the time of nucleosynthesis was $\epsilon = \alpha T_{nuc}^4 = 7.56 \times 10^{20}\,\mathrm{J\,m^{-3}}$, according to the law for the energy density for blackbody radiation (note: in reality we should multiply this energy density by the factor 1.681 to account for the neutrinos; but Gamow did not know much about the three families of neutrinos and their interactions in 1948).

   (b) Since the universe was radiation dominated, the critical density had to be equal to this radiation density, so $3H^2/(8\pi G) = \epsilon/c^2$. This gives $H = 2.17 \times 10^{-3}\,\mathrm{s^{-1}}$.

   (c) The age for the radiation dominated universe is $t = 1/(2H) = 231$ s.

   (d) For a present age $t_0 = 10^{10}$ years, the temperature is given by $3H_0^2/(8\pi G) = 3/(32\pi G t_0^2) = \alpha T_0^4$, which gives $T_0 = 27$ K. Note that actually this temperature just depends on t_0 and the assumption of a flat, radiation-dominated universe, but it does not depend on T_{nuc}.

   (e) If the universe changed from being radiation dominated to matter dominated at some redshift z_{rm}, then at the present time the matter density is greater than the radiation density by a factor $1 + z_{rm}$; so $\epsilon_r = \epsilon_m/(1 + z_{rm})$. In a flat universe with only matter and radiation, the total density has to be equal to the critical density, therefore $\epsilon_r + \epsilon_m = \epsilon_r(2 + z_{rm}) = \epsilon_{crit}$. So, $\epsilon_r = 3H_0^2/(8\pi G)/(2 + z_{rm}) = \alpha T_0^4$, and the radiation temperature is smaller by a factor $(2 + z_{rm})^{-1/4}$.

3. (a) The reason why heavy elements are synthesized in stars and not in the Big Bang is because stars live longer than the age of the universe when the temperature was

right for making nuclei, and they are denser than the matter in the universe was at this time.

TRUE. When the universe had a temperature comparable to the cores of red giants today (about 10^8 K), there was a mixture of hydrogen and helium in the universe, but carbon and heavier elements were not made because of the low baryon density and short time available compared to the interiors of red giants.

(b) For the Benchmark model, which includes a cosmological constant, the age of the universe at redshift $z = 10$ was shorter than in a flat space model containing only matter, with the same present Hubble constant.

FALSE. The cosmological constant implies an acceleration of the expansion rate at the present time, meaning that for a fixed present day Hubble constant, the expansion was slower in the past, so the age at a given redshift was longer. Alternatively, one can think of this by noting that the matter density today is smaller if part of the critical density is not matter, and then the matter density was also smaller at a fixed redshift in the past when the universe was matter dominated and the matter density was equal to the critical density, so the Hubble constant had to be smaller, and the age longer, at a fixed redshift.

(c) The Sachs-Wolfe effect is the process that generates Cosmic Microwave Background fluctuations from the Doppler effect due to the motion of the baryon-photon fluid in the last scattering surface.

FALSE. The Sachs-Wolfe effect is due to the fluctuations in the gravitational potential, and the fact that photons had to move in or out of the potential wells, being blueshifted or redshifted. The effects of the motion of the baryon-photon fluid is something different, which is part of the accoustic waves and causes anisotropies on smaller angular scales.

(d) More than half of all the helium that exists today in the Earth was created in nuclear reactions that occurred within the first five minutes after the Big Bang.

TRUE. During the first five minutes, about 25% of baryonic matter turned into helium. When the Sun was made, its composition was about 27% helium, so relatively little helium had been made by stars and ejected to the interstellar medium from which the Sun formed, compared to what was made in the Big Bang.

4. (a) The number density of photons per unit frequency is equal to the energy density per unit frequency divided by $h\nu$, or

$$n(\nu)\, d\nu = \frac{8\pi\nu^2\, d\nu}{c^3\left[\exp(h\nu/kT) - 1\right]} \ . \tag{2}$$

The total number density is found by integrating over frequency, which gives:

$$n = \frac{8\pi}{c^3} \int \frac{\nu^2\, d\nu}{\exp(h\nu/kT) - 1} \ . \tag{3}$$

Substituting $x = h\nu/(kT)$, we find

$$n = \frac{8\pi}{c^3} \left(\frac{kT}{h}\right)^3 \int \frac{x^2\, dx}{e^x - 1} \ . \tag{4}$$

For $T_0 = 2.725$, we find $n = 410.4\,\mathrm{cm}^{-3}$.

(b) The current density of baryons must then be $n_b = 410.4\,\mathrm{cm}^{-3} \cdot 5.5 \times 10^{-10} = 2.25 \times 10^{-7}\,\mathrm{cm}^{-3}$.

(c) Every baryon weighs approximately like the mass of a proton (this is not exact because, for example, the helium nuclei weigh a little less than 4 protons because of the helium nucleus binding energy, but the difference is rather small). So the density of baryons is $n_b m_p = 3.78 \times 10^{-31}\,\mathrm{g\,cm}^{-3}$. The critical density is $3H_0^2/(8\pi G) = 9.2 \times 10^{-30}\,\mathrm{g\,cm}^{-3}$, so $\Omega_b = 0.041$.

5. (a) The Hubble constant, in terms of the scale factor a normalized to $a_0 = 1$ today, is

$$H = H_0\sqrt{\Omega_{m0}/a^3 + \Omega_{\Lambda 0}}\ . \tag{5}$$

At $t \gg t_0$, $a \gg 1$ and $H \simeq H_0\sqrt{\Omega_{\Lambda 0}}$.

(b) We could attempt to solve the problem exactly by using equation (6.28) to relate time and scale factor. Instead, we can just note that at $t = 50t_0$, the universe will have expanded exponentially over many e-folding times, so essentially we can use the approximation $H = H_0\sqrt{\Omega_{\Lambda 0}}$ found in the previous question because $a \gg 1$. So the critical density will be

$$\epsilon_{crit} = \frac{3H^2c^2}{8\pi G} = 6.04 \times 10^{-8}\,\mathrm{erg\,cm}^{-3}\ . \tag{6}$$

The photon density will be equal to this critical density right after the vacuum energy has decayed.

(c) Using equations (6.28) and (6.27), we find first of all that the present age (when $a = 1$) is $H_0 t_0 = 2/3/\sqrt{1-\Omega_{m0}}A_0$, where

$$A_0 = \log(a_{m\Lambda}^{-3/2} + \sqrt{1 + a_{m\Lambda}^{-3}}) = 1.272\ , \tag{7}$$

where the numerical value is for $\Omega_{m0} = 0.27$. At $t = 50t_0$, $a \gg a_{m\Lambda}$, so we can approximate

$$50A_0 = \log[(a/a_{m\Lambda})^{3/2} + \sqrt{1 + (a/a_{m\Lambda})^3}] \simeq \log[2(a/a_{m\Lambda})^{3/2}]\ . \tag{8}$$

The solution for a is

$$a = a_{m\Lambda}\left(\frac{e^{50A_0}}{2}\right)^{2/3} = 1.18 \times 10^{18}\ . \tag{9}$$

The matter density will be the present density divided by a^3, so it is incredibly tiny. It will be smaller than the photon density by a factor

$$\frac{\epsilon_m}{\epsilon_{rad}} = \frac{\Omega_{m0}}{\Omega_{\Lambda 0}a^3} = 2.24 \times 10^{-55}\ . \tag{10}$$

Hence, right after the decay of the vacuum energy into photons, the photon density will be similar to the present critical density, because the vacuum energy has stayed constant until its decay. But the matter density will have been reduced to essentially nothing, due to the exponential expansion of the universe.

(d) The radiation will start decreasing as a relative to the matter, but because it is 4.5×10^{54} times larger than the matter just after the decay of the vacuum energy, the universe will have to expand by a factor $\sim 4.5 \times 10^{54}$ before matter can dominate again. During this radiation-dominated phase, the Hubble constant would change proportionally to a^{-2}, so it would have to decrease from its initial value $H_0\sqrt{\Omega_{\Lambda 0}}$ by a factor $(4.5 \times 10^{54})^2$, by which point the age of the universe would be $t \simeq 1/(2H) \sim 10^{119}$ years. It would indeed take a lot of expansion and a lot of time for matter to dominate again.