1. (a) The total mass inside \( R \) is obtained from \( GM(R)/R = v_c^2(R) \).

The answer can of course be found by substituting for the value of \( G \) and everything else in your favorite system of units, and if you are lucky not to make any mistake you may even get the right answer. Often, it is faster and safer to work it out using proportionality comparing to an example that you know and love. What could this example be but the Earth moving around the Sun? For the Earth, with \( 1M_\odot \) and an orbit of 1 AU the velocity is \( 30 \text{ km s}^{-1} \) (if you didn’t know how fast the Earth moves around the Sun, this is a good number to remember). So, the mass inside radius \( R \) is

\[
M(R) = \frac{\left(\frac{210 \text{ km s}^{-1}}{30 \text{ km s}^{-1}}\right)^2 8 \text{ kpc}}{1 \text{ AU}} M_\odot = 7^2 \cdot 8000 \cdot 206265M_\odot = 8 \times 10^{10}M_\odot. \tag{1}
\]

You needed to remember here that 1 pc = 206265 AU (which is also the number of arc seconds in a radian).

(b) If the density at \( R \) is \( \rho_0 \), then the density at any other radius \( r \) is \( \rho_0 (R/r)^2 \), so the mass inside \( R \) is

\[
M(R) = 4\pi \int_0^R dr r^2 \rho_0 \left(\frac{R}{r}\right)^2 = 4\pi \rho_0 R^3. \tag{2}
\]

Hence the density at \( R \) is

\[
\rho_0 = \frac{M(R)}{4\pi R^3} = 0.51m_p \text{ cm}^{-3}. \tag{3}
\]

The result is most easily computed remembering that the solar mass contains \( 1.19 \times 10^{57} \) proton masses (another useful number to remember), and a parsec is \( 3.086 \times 10^{18} \) cm.

(c) The density is

\[
\rho_\Lambda = \frac{3H_0^2}{8\pi G} = 5.5 \times 10^{-6}\Omega_\Lambda 0 m_p \text{ cm}^{-3} = 4 \times 10^{-6}m_p \text{ cm}^{-3} \tag{4}
\]

(d) Because the dark energy is spread out uniformly, whereas the dark matter and baryonic matter are highly concentrated in the inner parts of galaxies, the density of dark energy is very small compared to the density of matter inside the radius of the solar orbit in the Milky Way. The dark energy therefore must have a tiny dynamical effect.
2. The flux of one of these objects is \( f = L/(4\pi d_L^2) \), and its angular size is \( \theta = \ell/d_A \). Hence, the surface brightness is \( \Sigma \propto f/\theta^2 \), or

\[
\Sigma = \text{constant} \times \frac{f}{\theta^2} = \text{constant} \times \frac{d_A^2}{d_L^2}.
\] (5)

Note that \( L, \ell, \) and \( 4\pi \) are constants and so can be absorbed in the constant of proportionality. Since \( d_A = S_k(r)/(1 + z) \) and \( d_L = S_k(r)(1 + z) \), we have

\[
\Sigma = \text{constant} \times (1 + z)^{-4}.
\] (6)

Therefore, the surface brightness will always decrease with redshift as \((1 + z)^{-4}\) compared to the intrinsic surface brightness, without any dependence on the cosmological model.

3. The optical depth, for a flat model with a cosmological constant, is given by

\[
\tau(z) = \int n_e \sigma_e c \, dt = \frac{n_{e0} \sigma_e c}{H_0} \int_0^z \frac{(1 + z')^3 \, dz'}{(1 + z') \sqrt{\Omega_{m0}(1 + z')^3 + \Omega_{\Lambda0}}},
\] (7)

which has the result (doing the substitution \( y = \sqrt{\Omega_{m0}(1 + z')^3 + \Omega_{\Lambda0}} \))

\[
\tau(z) = \frac{2n_{e0} \sigma_e c}{3H_0 \Omega_{m0}} \left( \sqrt{\Omega_{m0}(1 + z)^3 + \Omega_{\Lambda0}} - 1 \right).
\] (8)

To find the redshift \( z \) at which \( \tau = 0.12 \), we need to calculate the present electron density \( n_{e0} \), which depends on \( \Omega_b0 \). Assuming all baryons are hydrogen, then \( n_{e0} = (\rho_{\text{crit},0}/m_H)\Omega_b0 = 2.53 \times 10^{-7} \), for \( \Omega_b0 = 0.045 \) and \( H_0 = 70\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1} \). Then the optical depth \( \tau = 0.12 \) is reached at redshift \( z = 11.5 \).

If we take into account that 24\% of the baryons are in the form of helium, then the 76\% of the baryons that are hydrogen still result in a fraction 0.76 of the electrons we obtained before, and the 24\% that are helium result in one electron for every two nucleons (because helium has four nucleons, and two electrons come with it to maintain charge neutrality), which gives an additional fraction of 0.12. Hence, the number of electrons is reduced by \( 1 - Y/2 = 0.88 \) relative to the case when all baryons are hydrogen, giving \( n_{e0} = 2.22 \times 10^{-7} \text{cm}^{-3} \). This implies that \( \tau = 0.12 \) is reached at \( z = 12.5 \).

4. (a) A component with equation of state \( p = w\epsilon \) will change its energy density according to the equation

\[
\frac{\dot{\epsilon}}{\epsilon} = -3(\epsilon + p) \frac{\dot{a}}{a} = -3(1 + w) \frac{\dot{a}}{a},
\] (9)

implying that, if \( \epsilon_0 \) is the present energy density, then \( \epsilon = \epsilon_0 (1 + z)^{3(1+w)} \).

In a flat universe, the present energy density has to be equal to the critical density, and since there is only one component we have:

\[
\epsilon_0 = \frac{3H_0^2c^2}{8\pi G}.
\] (10)
Using Friedmann’s equation for a flat universe, we find
\[ H^2(z) = \frac{8\pi G}{3c^2} \epsilon = H_0^2(1 + z)^{3(1+w)} , \] (11)

The comoving distance \( r \) is given by
\[ r = c \int_0^z \frac{dz}{H(z)} = \frac{c}{H_0} \int_0^z dz \frac{1}{(1 + z)^3(1 + 3w)} \left[ 1 - \frac{1}{(1 + z)^{1+3w}/2} \right] . \] (12)

(b) The angular diameter distance is
\[ d_A(z) = \frac{r}{1+z} = \frac{2c}{(1+3w)H_0(1+z)} \left[ 1 - \frac{1}{(1 + z)^{1+3w}/2} \right] . \] (13)

(c) The luminosity diameter distance is
\[ d_L(z) = r(1+z) = \frac{2c(1+z)}{(1+3w)H_0} \left[ 1 - \frac{1}{(1 + z)^{1+3w}/2} \right] . \] (14)

(d) To find the maximum of \( d_A(z) \), we find the root of its derivative:
\[ \frac{dd_A}{dz} = \frac{2c}{(1+3w)H_0} \left[ -\frac{1}{(1 + z)^2} + \frac{3+3w}{2} \frac{1}{(1 + z)^{(3+3w)/2}} \right] = 0 . \] (15)

The solution is \( (1 + z)^{(1+3w)/2} = (3 + 3w)/2 \), or
\[ z_{\text{max}} = \left( \frac{3 + 3w}{2} \right)^{2/(1+3w)} - 1 . \] (16)

(e) Substituting the above value of \( z \) into the expression for \( d_A \), we find that the maximum angular diameter distance is
\[ d_A(z_{\text{max}}) = \frac{2c}{H_0(1+3w)[(3+3w)/2]^{2/(1+3w)}} \left[ 1 - \frac{1}{(3+3w)/2} \right] = \frac{c}{H_0} \left( \frac{2}{3+3w} \right)^{3+3w/(1+3w)} . \] (17)