1. Friedmann’s equation is
\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G \epsilon(t)}{3c^2} - \frac{kc^2}{R_0^2 a^2(t)} .
\]  
(1)

For the case when the universe contains matter with negligible pressure only, the energy density changes as \( \epsilon(t) = \epsilon_0/a^3(t) \). Multiplying by \( a^2(t) \), we have
\[
(\dot{a})^2 = \frac{8\pi G \epsilon_0}{3c^2a} - \frac{kc^2}{R_0^2} .
\]  
(2)

Now, using \( \dot{a} = (da/dt) = (da/d\theta)/(dt/d\theta) \), we find that the left-hand-side of equation (2) is
\[
(\dot{a})^2 = \frac{c^2}{R_0^2} \frac{\sin^2 \theta}{(1 - \cos \theta)^2} = \frac{c^2}{R_0^2} \frac{1 + \cos \theta}{1 - \cos \theta} ,
\]  
(3)

where the last equality follows from \( \sin^2 \theta = 1 - \cos^2 \theta = (1 - \cos \theta)(1 + \cos \theta) \), and the right-hand-side of equation (2) is
\[
\frac{8\pi G \epsilon_0}{3c^2a} - \frac{kc^2}{R_0^2} = \frac{c^2}{R_0^2} \left( \frac{2}{1 - \cos \theta} - 1 \right) = \frac{c^2}{R_0^2} \frac{1 + \cos \theta}{1 - \cos \theta} .
\]  
(4)

So the two sides of equation (2) are indeed equal, confirming that this parametric solution given as \( a(\theta) \) and \( t(\theta) \) is indeed a solution of Friedmann’s equation.

(a) The maximum value of \( a \) occurs at \( \theta = \pi \), and is
\[
a_{\text{max}} = \frac{8\pi G \epsilon_0 R_0^2}{3c^4} .
\]  
(5)

(b) Correspondingly, the maximum value of the proper radius of curvature is
\[
a_{\text{max}} R_0 = \frac{8\pi G \epsilon_0 R_0^3}{3c^4} .
\]  
(6)

(c) The age of the universe at \( \theta = \pi \) is
\[
t_{\text{max}} = \frac{4\pi^2 G \epsilon_0 R_0^2}{3c^5} .
\]  
(7)

(d) The Big Crunch happens when \( \theta = 2\pi \), and we then have
\[
t_{\text{crunch}} = \frac{8\pi^2 G \epsilon_0 R_0^3}{3c^5} .
\]  
(8)
2. (a) For the model with $\Omega_{m0} = 1$, the comoving distance is

$$r = c \int_0^z \frac{dz}{H(z)} = \frac{c}{H_0} \int_0^z \frac{dz}{(1 + z)^{3/2}} = \frac{2c}{H_0} \left(1 - \frac{1}{\sqrt{1 + z}}\right).$$

(9)

The comoving distance to the horizon, at $a = 0$ or $z = \infty$, is $r = 2c/H_0$.

(b) For this model, half the comoving distance to the horizon is $r = c/H_0$, and the redshift at which the comoving distance has this value is obtained as:

$$1 - \frac{1}{\sqrt{1 + z}} = \frac{1}{2}; \quad z = 3.$$

(10)

(c) For this same model, and at $z = 3$, the age of the universe is obtained from

$$t(z) = \int_1^\infty \frac{dz}{(1 + z)H(z)} = \frac{2}{3H_0} \frac{1}{(1 + z)^{3/2}}.$$

(11)

The present age of the universe is of course $t_0 = 2/(3H_0)$, and so the ratio of the age at $z = 3$ to its present age is just

$$\frac{t(z = 3)}{t_0} = \frac{1}{(1 + z)^{3/2}} = \frac{1}{8}.$$

(12)

(d) From the same equation as above, we find

$$\frac{t(z)}{t_0} = \frac{1}{(1 + z)^{3/2}} = \frac{1}{2}; \quad z = 2^{2/3} - 1 = 0.5874.$$

(13)

Note that all these equations are of course valid only for the specific model that is flat and contains only matter, with $\Omega_{m0} = 1$.

3. For the benchmark model, which is flat and contains matter and a cosmological constant (or a component with $p = -\rho$), we can use equation (6.28) in the textbook.

First, we find from equation (6.27) the constant

$$a_{m\Lambda}^3 = \frac{\Omega_{m0}}{1 - \Omega_{m0}} = 0.370,$$

(14)

for $\Omega_{m0} = 0.27$. This is the scale factor at which the matter and cosmological constant energy densities are equal, using the normalization $a_0 = 1$ for the present.

The present age of the universe is obtained from equation (6.27) for $a = a_0 = 1$, and is

$$H_0 t_0 = \frac{2}{3\sqrt{1 - \Omega_{m0}}} \log A_0,$$

(15)

where we have defined

$$A_0 = a_{m\Lambda}^{-3/2} + \sqrt{1 + a_{m\Lambda}^3} \simeq 3.57.$$

(16)
Now, we want to find the value of $a$ for which $t = t_0/2$, as obtained from equation (6.27). This implies
\[ \log A = \frac{\log A_0}{2}, \]  
(17)

where
\[ A = x + \sqrt{1 + x^2}; \quad x = \left( \frac{a}{a_{m\Lambda}} \right)^{3/2}. \]  
(18)

The above equations imply $A = \sqrt{A_0}$, and so
\[ x + \sqrt{1 + x^2} = \sqrt{A_0}; \quad 1 + x^2 = (\sqrt{A_0} - x)^2; \quad x = \frac{A_0 - 1}{2\sqrt{A_0}} = 0.680, \]  
(19)

and
\[ a = a_{m\Lambda} x^{2/3} = 0.555; \quad z = \frac{1}{a} - 1 = 0.802. \]  
(20)

We see that the redshift at which the age of the universe was half the present age is larger in this benchmark model than in the model with $\Omega_{m0} = 1$. This is because in the benchmark model, which contains vacuum energy, the universe has started to accelerate recently, roughly since the epoch at $a = a_{m\Lambda}$. The universe took a longer time to expand to $a = 0.555$ and then it picked up speed again in its expansion up to the present $a_0 = 1$.

To prove equation (6.28) in the textbook, we start with the usual equation for the age of the universe at any redshift $z$, which in the case of the flat model with matter and a cosmological constant is
\[ t(z) = \frac{1}{H_0} \int_{z}^{\infty} \frac{dz}{(1 + z)\sqrt{\Omega_{m0}(1 + z)^3 + \Omega_{\Lambda0}}}, \]  
(21)

After substituting
\[ y = \sqrt{1 + \frac{\Omega_{m0}}{\Omega_{\Lambda0}}(1 + z)^3}, \]  
(22)

we find $2y dy = 3(y^2 - 1)dz/(1 + z)$, and so
\[ t = \frac{2}{3H_0\sqrt{\Omega_{\Lambda0}}} \int_{y}^{\infty} \frac{dy}{y^2 - 1}. \]  
(23)

The integral can be solved analytically as:
\[ -\int \frac{dy}{y^2 - 1} = \frac{1}{2} \log \frac{y + 1}{y - 1} = \log \frac{y + 1}{\sqrt{y^2 - 1}}, \]  
(24)

which yields for $t$
\[ t = \frac{2}{3H_0\sqrt{\Omega_{\Lambda0}}} \left[ \log \left( \frac{1}{\sqrt{y^2 - 1}} + \frac{y}{\sqrt{y^2 - 1}} \right) \right]_{\infty}^{1 + \Omega_{m0}(1 + z)^3/\Omega_{\Lambda0}}, \]  
(25)
\[ t = \frac{2}{3H_0 \sqrt{\Omega_\Lambda}} \log \left( \sqrt{\Omega_\Lambda \Omega_{m0}(1 + z)^3} + \sqrt{1 + \frac{\Omega_\Lambda}{\Omega_{m0}(1 + z)^3}} \right). \]  

(26)