1. At the final time of the invasion, \( t \), the invaders are at proper distance \( d_p \), and therefore comoving distance \( r = d_p/a(t) \).

(a) For a flat space, the proper volume is obviously the usual one in Euclidean geometry,

\[
V = \frac{4\pi}{3} d_p^3. \tag{1}
\]

(b) In a closed universe, and if \( R \) is the comoving radius of curvature, the proper area of a sphere at comoving coordinate \( r \) is \( 4\pi a^2(t) R^2 \sin^2(r/R) \), and the proper distance between two spheres at \( r \) and \( r + dr \) is just \( a(t) \, dr \), as obtained from the Friedmann-Robertson-Walker metric. Therefore, the proper volume of each spherical shell between \( r \) and \( r + dr \) is \( 4\pi a^3(t) R^2 \sin^2(r/R) \, dr \), and the proper volume of the sphere is

\[
V = 4\pi a^3(t) R^2 \int_0^r \sin^2(r/R) \, dr = 4\pi a^3(t) R^3 \int_0^{r/R} \sin^2(r/R) \, d(r/R) . \tag{2}
\]

which gives (remembering that \( r = d_p/a \))

\[
4\pi a^3 R^3 \left( \frac{d_p}{2aR} - \frac{\sin(2d_p/aR)}{4} \right) . \tag{3}
\]

(c) In an open universe, the calculation is just like for the closed universe but with the sinh function:

\[
V = 4\pi a^3(t) R^2 \int_0^{r/R} \sinh^2(r/R) \, d(r/R) = 4\pi a^3 R^3 \left( \frac{\sinh(2d_p/aR)}{4} - \frac{d_p}{2aR} \right) . \tag{4}
\]

2. (a) The redshift of a galaxy is given by \( z = H_0 r/c \), where \( r \) is its distance, as long as the redshift is small compared to one and peculiar velocities (departures from the Hubble flow) are negligible:

TRUE. Hubble’s law says \( v = H_0 r \), and the redshift is \( z = v/c \) when the velocity is small compared to the speed of light, or when the redshift is \( z \ll 1 \).

(b) If a traveler leaves the Earth at half the speed of light, reaches out to a distance of half a light-year during a year of travel, then turns around instantaneously and comes back to the Earth at the same constant speed (arriving 2 years after the departure according to the people in the Earth), the watch of the traveler will show that the trip has taken only a time \((2/\sqrt{3})\) years.

FALSE. The time of the traveler’s watch \( t' \) is related to the time measured at the Earth, \( t \), by \( t' = t/\gamma \), where \( \gamma = 1/\sqrt{1 - v^2/c^2} = 2/\sqrt{3} \). So, the watch of the traveler will indicate \( \sqrt{3} \) years.
(c) If the universe is open, triangles in space at a fixed cosmic time could have the sum of their three angles be either greater or smaller than $\pi$.

FALSE. An open universe is negatively curved, which implies that all triangles have the sum of their three angles adding to less than $\pi$ (although the difference with $\pi$ becomes very small and hard to notice for triangles that are much smaller than the radius of curvature).

(d) If the redshift of a galaxy is 2, it means that at the cosmic time when the light we see now was emitted, the proper distance to this galaxy was three times smaller than the present one.

TRUE. Scale factor and redshift are related by $a = 1/(1 + z)$, if the scale factor is normalized to unity at the present time.

3. Cosmic time is the proper time measured by an observer who is comoving with the expansion of the universe, starting at $t = 0$ in the Big Bang, at every point in space at fixed comoving coordinates.