Solution to the Midterm Exam, May 2 2005

1. For each one of the following statements, say whether they are true or false, and write a brief explanation (usually only one sentence) that says why it is true or false.

   (a) If a galaxy that has no peculiar velocity emits a photon at wavelength λ_e at the epoch corresponding to redshift z, and we observe the photon today at wavelength λ_{obs}, it is always exactly true that $\lambda_{obs} - \lambda_e = z\lambda_e$, no matter how large z may be and no matter what cosmological model we are considering.

   TRUE. This is the definition of redshift, and it is applicable for any value of z.

   (b) In Newton's theory of gravity, one can understand the evolution of the scale factor of the universe by considering a finite sphere cut out from the universe, and one can already see in this theory that there are models in which the expansion of the universe can accelerate.

   FALSE. Although it is true that Newton's theory yields the correct equation for the evolution of the scale factor when considering a finite sphere cut out from the universe when there is only matter, Newton's theory predicts that the expansion should always decelerate. It is only general relativity that predicts that all other types of energy must be included in the total energy density, that pressure also affects the second time derivative of the scale factor, and that as a result the scale factor may accelerate when a large negative pressure is present.

   (c) For all cosmological models, the present age of the universe is equal to $1/H_0$, where H_0 is the present Hubble constant.

   FALSE. This is only true for an empty model, an open model with no energy density.

   (d) A trip from the Solar System to the center of the galaxy (25000 light-years away from the Sun) could in principle take only 25 years according to the watch of the traveller, if the traveller were moving as fast as 0.999 times the speed of light.

   FALSE. Even though the time measured in the spaceship is indeed shorter, in order to be reduced by a factor of 1000 it would have to travel at a velocity that gives a Lorentz factor $\gamma = (1 - v^2/c^2)^{-1/2} = 1000$, or $v = c(1 - \gamma^{-2})^{1/2} \simeq 0.9999995c$.

   (e) The equivalence principle says that no experiment can distinguish a frame of reference that is in free-fall from a frame of reference in flat space-time (that is to say, with no gravity) that does not accelerate in any way, as long as tidal accelerations in the free-falling frame are not important.

   TRUE. There is indeed no way to distinguish a free-falling frame from a frame that does not accelerate in the absence of gravity, except from measuring tidal accelerations. This is exactly why astronauts inside the space shuttle do not feel Earth's gravity, and why we don't feel the gravity of the Sun. The equivalence principle also says that a frame at rest on the surface of the Earth cannot be distinguished from a frame that is accelerated by g in the absence of gravity.

(f) The metric $ds^2 = dr^2 + r^2 d\theta^2$ is that of a two-dimensional space with zero curvature; in other words, it is the metric of a flat two-dimensional space.

TRUE. This is the metric obtained in polar coordinates, by doing the substitution $r = x^2 + y^2$ and $\tan\theta = y/x$, starting from cartesian coordinates in a flat 2-dimensional space.

(g) If the universe is flat, two galaxies that are both at a comoving distance r from us and with an angular separation α between them on the sky (in other words, the directions from us to each galaxy subtend an angle α) are at a proper distance $2a(t)r\sin(\alpha/2)$ from each other, where $a(t)$ is the scale factor.

TRUE. In a flat universe, we apply regular trigonometry to figure out that the distance x between the two galaxies obeys $x/2 = a(t)r\sin(\alpha/2)$.

(h) If the universe is closed, two galaxies that are both at a comoving distance r from us and with an angular separation α between them on the sky are at a proper distance from each other that is less than $2a(t)r\sin(\alpha/2)$, for all cosmological models as long as the space geometry is closed.

TRUE. Geodesics emerging from the origin get closer to each other as the distance increases, compared to flat space. This is just because in the closed model, all the distances perpendicular to the direction of increasing r are shortened by the factor $R\sin(r/R) < r$. In a two-dimensional globe, we see how two geodesics coming out of the pole eventually start approaching each other, making the distance between them shorter than it would be in flat space. Similarly, in the example of the soldiers that go out to invade the universe, the soldiers find themselves getting closer to each other even though they all move away from the point they started from along a geodesic.

(i) A galaxy seen at a redshift of 2 is observed at the cosmic time when the universe had a mean density of matter that was 27 times larger than today, in all cosmological models.

TRUE. The density declines in proportion to $(1+z)^3$ with time, so at $z = 2$ it was 27 times larger than today.

(j) The energy density of radiation declines more slowly than the energy density of matter as the universe expands.

FALSE. The radiation drops as $(1+z)^4$, faster than the matter which drops as $(1+z)^3$.

2. Define geodesic.

Geodesic is the path between two points in any space that has the shortest distance, as calculated from the metric. In a flat space, geodesics are straight lines.

(The above definition is all that was required. A further detail is that, in curved space-time, a particle will move along a geodesic which is the path that has the *maximum* proper time between two events, in this case it is a maximum because of the minus sign that appears in the metric for the space coordinates. In general a geodesic must have a proper distance or proper time between two events that is an extremum.)

3. (a) Imagine a universe with a flat space metric that is filled only with radiation (and no matter or any other component), with the spectrum of a blackbody. At the time when the radiation has a temperature $T = 3$ K, what would be the energy density in the universe?

The energy density of blackbody radiation is always $\epsilon = \alpha T^4$, or for $T = 3$ K, $\epsilon = 6.13 \times 10^{-13}\,\mathrm{erg\,cm^{-3}}$.

(b) At this time when $T = 3$ K, and remembering again that the space metric is assumed to be flat and that the *only* energy density is the radiation, what would be the Hubble constant? Find the answer first in $\mathrm{s^{-1}}$, and then convert it to units of $\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

Because space is flat, the total energy density (which in this case is just the energy density of the blackbody radiation) must be equal to the critical value:

$$\epsilon = \frac{3H^2c^2}{8\pi G} \; ; \qquad H = \sqrt{\frac{8\pi G\epsilon}{3}} \cdot \frac{1}{c} = 1.95 \times 10^{-20}\mathrm{s^{-1}} = 0.6\,\mathrm{km\,s^{-1}\,Mpc^{-1}} \; . \qquad (1)$$

(c) What would be the age of the universe (cosmic time t) when the radiation has a temperature $T = 3$ K (remember again: it is a flat universe that contains *only* radiation)?

For a flat universe dominated by radiation, we have seen that the age is

$$t = \frac{1}{2H} = 2.56 \times 10^{19}\,\mathrm{s} = 8.1 \times 10^{11}\,\mathrm{years} \qquad (2)$$

Note that it is not surprising that in this universe, the age would have to be much longer than in our present universe. If the energy density of blackbody radiation with similar temperature as the present blackbody radiation needs to account by itself for all the mass-energy density in the universe, it means that the universe must have expanded a lot more, for a lot longer time, than our present universe, to reach this very low energy density.

(d) What would be the age of the universe when the radiation has some other temperature T? Write down the answer for the age t as a function of T.

$$t = \frac{1}{2H} = \frac{c}{2T^2} \cdot \sqrt{\frac{3}{8\pi G\alpha}} \; . \qquad (3)$$