Solution to the Final Exam, June 8 2005

1. (a) For any model of the universe in which space is flat, and if r is the comoving distance to an object, the angular diameter distance is always equal to $r/(1+z)$, independently of the abundances of matter, radiation, and vacuum energy.

   TRUE. The angular diameter distance is $r/(1+z)$ when space is flat ($S_k(r) = r$ in the metric).

   (b) The linear density fluctuations in the dark matter did not start growing until after the epoch of recombination.

   FALSE. The dark matter density fluctuations were already growing, although slowly, during the radiation dominated era, and then they started growing in proportion to the scale factor after the era of matter-radiation equality at $z \simeq 3500$, before recombination at $z \simeq 1100$.

   (c) If the sum of the densities of all matter, radiation and vacuum energy in the universe is greater than the critical density, then (assuming the validity of General Relativity and the cosmological principle) space must have the geometry of a three-dimensional spherical surface.

   TRUE. When the total energy density is greater than the critical density, the universe must be closed, which implies that space has a constant positive curvature, giving it the geometry of a three-dimensional spherical surface.

   (d) In a cluster of galaxies, the velocity dispersion of the galaxies is about the same as the average velocity dispersion of the particles in the hot, X-ray emitting gas.

   TRUE. Both the galaxies and the plasma are in dynamical equilibrium in the same gravitational potential well.

   (e) At the time of nucleosynthesis, when the temperature of the radiation was $\sim 3 \times 10^8$ times larger than today, the density of baryons in the universe was also 3×10^8 times larger than today.

   FALSE. Whereas the temperature of the radiation scales as the inverse of the scale factor, the matter density scales as the inverse cube of the scale factor, so the density of baryons would be larger by a factor $(3 \times 10^8)^3$.

   (f) General Relativity predicts that a ray of light passing through a cluster of galaxies will be deflected by an angle equal to that predicted by Newton's theory, if one assumes that light is made of particles travelling at a speed c.

   FALSE. General Relativity predicts a deflection angle twice as large as Newton's theory when applied to a particle moving at a speed c, whenever this deflection angle is small (compared to 1 radian).

(g) If the universe were open, the first peak in the Δ_l spectrum of the Cosmic Microwave Background anisotropies would occur at smaller values of l than is observed.

FALSE. If the universe is open, the angular diameter distance to the epoch of recombination is increased, which means that the angular size of the horizon at the epoch of recombination is smaller. This shifts the characteristic scale of the peak of Δ_l to smaller angles, implying larger l.

(h) The reason why primordial deuterium has a fractional abundance as low as 3×10^{-5} is that neutrons decayed too fast, and so not much deuterium could ever be made.

FALSE. As deuterium is made, it rapidly fuses further to make helium. This is why very little deuterium is left from primordial nucleosynthesis. Fewer than half of the neutrons that are left after freeze-out decay, and the majority result in helium production.

(i) If the reaction

$$p + e^- \qquad n + \nu_e \ . \tag{1}$$

had a much larger cross section (i.e., occurred at a much larger rate), therefore allowing the proton and neutron abundances to remain in thermodynamic equilibrium until a later time, much more helium would have been made in the Big Bang.

FALSE. If this reaction proceeded faster, freeze-out of the neutrons would occur later, at a lower temperature, implying a much lower neutron abundance because of the Boltzmann factor. With many fewer neutrons left, much less helium would be made.

(j) Whereas in the Big Bang model with an initial singularity, distant regions of the universe that we observe today on the Cosmic Microwave Background were never in communication with each other in the past, in the inflationary model all such regions were previously in communication.

TRUE. These distant regions in the universe were in communication in the past during the inflationary epoch.

2. (a) The energy density was

$$\epsilon = \alpha T^4 = 7.56 \times 10^{21}\,\mathrm{erg\,cm^{-3}} \ . \tag{2}$$

(b) The scale factor compared to the present time was $a = T_0/T = 2.7 \times 10^{-9}$. The baryonic density at present is:

$$\rho_{b0} = \frac{3H_0^2\Omega_{b0}}{8\pi G} = 4.14 \times 10^{-31}\,\mathrm{g\,cm^{-3}} \ , \tag{3}$$

So the density of baryons when $T = 10^9\,\mathrm{K}$ was

$$\rho_b = \rho_{b0}/a^3 = 2.05 \times 10^{-5}\,\mathrm{g\,cm^{-3}} \ . \tag{4}$$

(c) The rest-mass energy of baryons when $T = 10^9\,\mathrm{K}$ was $\rho_b c^2 = 1.84 \times 10^{16}\,\mathrm{erg\,cm^{-3}}$. Multiplying by the efficiency of nuclear fusion to helium, 0.007 and dividing by 4 (since only one quarter of the matter undergoes the reaction), we obtain that the density of nuclear fusion energy released is $3.22 \times 10^{13}\,\mathrm{erg\,cm^{-3}}$.

(d) The energy released by the fusion is more than eight orders of magnitude smaller than all the radiation energy in the universe. The radiation was not much affected.

3. (a) Friedman's equation, applied at the present time (for $k = -1$), says

$$\frac{\dot{a}^2}{a^2} = H_0^2 = \frac{8\pi G\rho}{3} + \frac{c^2}{R^2} = \Omega_{m0} H_0^2 + \frac{c^2}{R^2} \ . \tag{5}$$

This implies that

$$R = \frac{c}{H_0\sqrt{1-\Omega_{m0}}} \ . \tag{6}$$

(b) Substituting for $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, and $\Omega_{m0} = 0.3$, we find $R = 5120$ Mpc.

(c) The comoving distance is

$$r = \int_0^\infty \frac{c\,dz}{H(z)} = \frac{c}{H_0}\int_0^\infty \frac{dz}{\sqrt{\Omega_{m0}(1+z)^3 + (1-\Omega_{m0})(1+z)^2}} \ , \tag{7}$$

$$r = \frac{c}{H_0}\int_0^\infty \frac{dz}{(1+z)\sqrt{1+\Omega_{m0}z}} = \frac{c}{H_0\sqrt{1-\Omega_{m0}}} \log\frac{1+\sqrt{1-\Omega_{m0}}}{1-\sqrt{1-\Omega_{m0}}} \ . \tag{8}$$

(d) The product $d_A(z)\cdot(1+z)$ is:

$$d_A(z)\cdot(1+z) = R\sinh(r/R) = R\sinh\left(\log\frac{1+\sqrt{1-\Omega_{m0}}}{1-\sqrt{1-\Omega_{m0}}}\right) \ , \tag{9}$$

which yields

$$d_A(z)\cdot(1+z) = \frac{R}{2}\left(\frac{1+\sqrt{1-\Omega_{m0}}}{1-\sqrt{1-\Omega_{m0}}} - \frac{1-\sqrt{1-\Omega_{m0}}}{1+\sqrt{1-\Omega_{m0}}}\right) \ , \tag{10}$$

$$d_A(z)\cdot(1+z) = \frac{R}{2}\,\frac{4\sqrt{1-\Omega_{m0}}}{1-(1-\Omega_{m0})} = \frac{2R\sqrt{1-\Omega_{m0}}}{\Omega_{m0}} = \frac{2c}{H_0\Omega_{m0}} \ . \tag{11}$$